Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 7, 2004*
Revised: *November 1, 2004*
Accepted: *November 2, 2004*

One-loop maximal helicity violating amplitudes in $N = 4$ super Yang-Mills theories

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Abstract: One-loop maximal helicity violating (MHV) amplitudes in $N = 4$ super Yang-Mills (SYM) theories are analyzed, using the prescription of Cachazo, Svrček, and Witten (CSW). The relations between leading N_c amplitudes $A_{n;1}$ and sub-leading amplitudes $A_{n;c}$ obtained by the CSW prescription are found to be identical to those obtained from conventional field theory calculations. Combining with existing results, this establishes the validity of the CSW prescription to one-loop in the calculation of MHV amplitudes in $N = 4$ SYM theories of finite N_c.

Keywords: Gauge Symmetry, Extended Supersymmetry, QCD.

Contents

1. Introduction

In perturbative quantum field theories, the Feynman diagram technique provides a standard method for calculating scattering amplitudes, but not necessary an efficient one. In practice, the method could be tedious and may require huge amount of calculations. On the other hand, the end results of most calculations are much simpler than the intermediate steps. The simplification can be drastic if the theory is constrained by symmetries, though the calculation could instead be involved due to the proliferation of particles and couplings. For example, the Parke-Taylor formula for maximal helicity violating (MHV) amplitudes in Yang-Mills theories can be put within a single line which summarizes enormous number of Feynman diagrams [1]. The simplification goes further if the theory is constrained by extra symmetries. For $N = 4$ supersymmetric Yang-Mills (SYM) theories, one-loop results also assume extremely simple forms [2, 3].

One is thus naturally led to search for new methods for specific theories which would take the symmetries into account in the first place. In the specific case of Yang-Mills theories, various techniques were developed and tremendous progress has been achieved. Among them, the color decomposition method and the spinor helicity technique [4] have been proved to be extremely useful and efficient [5].

Recently, Witten pointed out a deep relation [6] between $N = 4$ SYM theories and one type B topological string theory, by re-expressing SYM scattering amplitudes in the language of twistor theories [7]. Unfortunately, this relation breaks down beyond tree level, as the supergravity part in the string construction does not decouple form the SYM part [8]. However, taking advantage of insights thus gained and by a careful analysis of known helicity amplitudes, Cachazo, Svrček, and Witten (CSW) [9] proposed a novel prescription to calculate tree level amplitudes, which uses the MHV amplitudes as vertices to construct all other amplitudes. The efficiency of the method is phenomenal and the validity of the method has been checked by various tree level calculations [10]–[18]. In [19], the method has been extend to the calculation of one-loop MHV amplitudes and the validity of the method is reconfirmed in the large N_c limit. The twistor-space structure of one-loop amplitudes are further studied in [20, 21]. On the other hand, tree-level amplitudes were also obtained from connected curves in twistor string theories [22, 23, 24].

In this paper, one-loop MHV amplitudes in general $N = 4$ SYM theories are calculated beyond the large N_c limit, by including all other diagrams which are not survived at the large N_c limit. We reproduce the same relation between leading N_c amplitudes $A_{n;1}$ and sub-leading amplitudes $A_{n;c}$ obtained by the CSW prescription as those from conventional field theory calculations [2]. Combining with results in [19], this establishes the validity of the CSW approach to one-loop in the calculation of MHV amplitudes in SYM theories, without taking the large N_c limit. The CSW prescription is motivated by a string construction and the string construction so far seems mainly to be related to the large N_c limit. The sub-leading results obtained in this paper indicate a wider applicability. This calls for further inquiry of the rationale behind the prescription.

The paper will be organized as follows. In section 2, we review one-loop results from conventional field theory methods and the CSW prescription. In section 3, we analyze one-loop MHV amplitudes of four and five external particles by using the CSW prescription. In section 4, the analysis is repeated for arbitrary number of external particles. We conclude in section 5.

2. Review of existing one-loop results and the CSW prescription

In four dimensional space-time, a momentum k_μ can be expressed as a bispinor $k_{a\dot{a}} = k_\mu \sigma^\mu_{a\dot{a}}$. For massless particle, $k^2 = 0$, the momentum can be factorized $k_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ in terms of spinors λ_a, $\tilde{\lambda}_{\dot{a}}$ of positive and negative chirality. Spinor products are defined to be $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b$ and $\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \rangle = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$, which are usually abbreviated as $\langle 1, 2 \rangle$ and $[1, 2]$.

At tree level, the scattering amplitudes of n gluons with one or none opposite type of helicity vanish. The amplitudes with two negative helicity are called maximally helicity violating (MHV) amplitudes. For $N = 4$ SYM theories, a MHV amplitude is given by the generalized Parke-Taylor formula which includes particles of all helicity [25]:

$$i(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left(\sum_{i=1}^{n} \lambda_i \eta^i \right) A_n \left(\{k_i, \lambda_i, a_i\} \right) \tag{2.1}$$

where η^i_A are anti-commuting variables, A is an index of the anti-fundamental representation of SU(4); k_i, λ_i, and a_i are the momentum, helicity, and the color index of the i-th external particles, respectively;

$$A_n \left(\{k_i, \lambda_i, a_i\} \right) = \sum_{\sigma \in S_n / Z_n} \mathrm{Tr} \left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}} \right) \prod_{i=1}^{n} \frac{1}{\langle \sigma(i), \sigma(i+1) \rangle} \tag{2.2}$$

where S_n / Z_n is the set of non-cyclic permutation of $\{1, \ldots, n\}$. The U(N_c) generators T^a are the set of hermitian $N_c \times N_c$ matrices normalized such that $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. Here and after the gauge coupling constant g is not included, but can be easily recovered when needed. The supersymmetric amplitudes can be expended in powers of the η_A^i, and each term of this expression corresponds to a particular scattering amplitude.

To one-loop, the results take the form [2]

$$A_n^{1-\mathrm{loop}}\left(\{k_i,\lambda_i,a_i\}\right) = \sum_{c=1}^{[n/2]+1} \sum_{\sigma\in S_n/S_{n;c}} Gr_{n;c}(\sigma) A_{n;c}(\sigma) \tag{2.3}$$

where S_n is the set of all permutation of n objects and $S_{n;c}$ is the subset leaving $Gr_{n;c}$ invariant. The color factors are

$$\begin{aligned} Gr_{n;1} &= N_c \,\mathrm{Tr}\left(T^{a_1}\cdots T^{a_n}\right) \\ Gr_{n;c} &= \mathrm{Tr}\left(T^{a_1}\cdots T^{a_{c-1}}\right)\mathrm{Tr}\left(T^{a_c}\cdots T^{a_n}\right) \end{aligned} \tag{2.4}$$

The simplest kinematic factors are proportional to the corresponding tree level ones [2, 3]

$$A_{n;1}(1,\ldots,n) = V_n \prod_{i=1}^{n} \frac{1}{\langle i,i+1\rangle} \tag{2.5}$$

where V_n is an universal function dependent of external momenta but not of the helicities. $A_{n;c>1}$ can be obtained by performing appropriate sums over permutations of $A_{n;1}$:

$$A_{n;c}(1,\ldots,c-1;c,\ldots n) = (-)^{c-1} \sum_{\sigma\in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma) \tag{2.6}$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1,\ldots,1\}$, $\beta_i \in \{\beta\} \equiv \{c,c+1,\ldots,n\}$ and $COP\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,\ldots,n\}$ with n held fixed that preserve the cyclic ordering of α_i within $\{\alpha\}$ and of β_i within $\{\beta\}$, while allowing for all possible relative ordering of α_i with respect to β_i.

In [9], it was proposed that these MHV amplitudes can be continued to off-shell and be used as vertices to generate tree diagrams for all non-MHV amplitudes. These new diagrams are dubbed as MHV diagrams. Including all relevant factors, the n-points MHV vertex is

$$\mathcal{V}_n = i\,\mathrm{Tr}\left(T^{a_1}\cdots T^{a_n}\right)(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{n}\lambda_i\tilde{\lambda}_i\right)\delta^{(8)}\left(\sum_{i=1}^{n}\lambda_i\eta^i\right)\prod_{i=1}^{n}\frac{1}{\langle i,i+1\rangle}\,. \tag{2.7}$$

The propagator for internal lines of momentum k is simply taken as i/k^2. The issue is how to define the corresponding spinor for particles with off-shell momentum k. The CSW prescription is to take an arbitrary spinor of positive helicity $\tilde{\xi}$ and define $\lambda_a = k_{a\dot{a}}\tilde{\xi}^{\dot{a}}$.

Extending these rules to one loop and to the leading order of N_c, the correct expression of $A_{n;1}$ is reproduced [19]. For example, the MHV diagram in figure 1 gives

$$A = i(2\pi)^4\delta^{(4)}(P_L+P_R)\int\frac{d^4L_1}{L_1^2}\int\frac{d^4L_2}{L_2^2}\int d^4\eta d^4\eta'\,\delta^{(4)}\left(L_2+L_1+P_L\right)A_L A_R \tag{2.8}$$

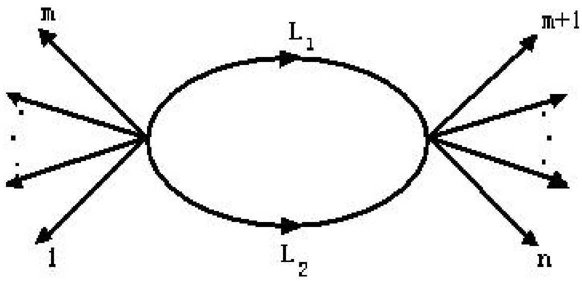

Figure 1: One leading order one-loop MHV diagram, using MHV amplitudes as interaction vertices, with the CSW off-shell prescription.

where P_L is the sum of momenta on the left of the diagram. Other factors are,

$$A_L = \mathrm{Tr}\left(T^a T^{a_1} \cdots T^{a_m} T^b\right) \delta^{(8)}(\Theta^L) \frac{1}{\langle l_2, 1\rangle\langle m, l_1\rangle\langle l_1, l_2\rangle} \prod_{i=1}^{m-1} \frac{1}{\langle i, i+1\rangle} \tag{2.9}$$

$$A_R = \mathrm{Tr}\left(T^b T^{a_{m+1}} \cdots T^{a_n} T^a\right) \delta^{(8)}(\Theta^R) \frac{1}{\langle -l_1, m+1\rangle\langle n, -l_2\rangle\langle -l_2, -l_1\rangle} \prod_{i=m+1}^{n-1} \frac{1}{\langle i, i+1\rangle}$$

where

$$\begin{aligned} \Theta^L_{A\alpha} &= \sum_{i\in L} \eta^i_A \lambda^i_\alpha + \eta'_A l_{2\alpha} + \eta_A l_{1\alpha} \\ \Theta^R_{A\alpha} &= \sum_{i\in R} \eta^i_A \lambda^i_\alpha - \eta'_A l_{2\alpha} - \eta_A l_{1\alpha} \end{aligned} \tag{2.10}$$

and up to irrelevant overall factors,

$$l_\alpha = L_{\alpha\dot\alpha} \tilde{\xi}^{\dot\alpha}, \qquad \tilde{l_{\dot\alpha}} = \xi^\alpha L_{\alpha\dot\alpha}.$$

Integrating out η and η', the fermionic part gives an overall $\delta^{(8)}(\sum_i^n \eta^i \lambda^i)$, which will be ignored in the rest of the paper. Summing up the color indices, we get the rest expression,

$$A = Gr_{n;1} L_n \prod_{i=1}^{n} \frac{1}{\langle i, i+1\rangle} \tag{2.11}$$

where

$$L_n = \int \frac{d^4 L_1}{L_1^2} \int \frac{d^4 L_2}{L_2^2} \delta^{(4)}(L_2 + L_1 + P_L) \frac{\langle 1, n\rangle\langle l_1 l_2\rangle}{\langle n, l_1\rangle\langle l_1, 1\rangle} \frac{\langle m, m+1\rangle\langle l_1 l_2\rangle}{\langle m, l_2\rangle\langle l_2, n\rangle}. \tag{2.12}$$

After an extensive calculation, L_n is shown to be independent of $\tilde{\xi}$. Including all inequivalent diagrams and summing up corresponding L_n's, the exact expression V_n in eq. (2.5) is reproduced. That is, the leading contribution in eq. (2.3) is recovered, and the validity of the CSW prescription established in the large N_c limit.

We now classify the one-loop MHV diagrams according to their topologies. One-loop MHV diagrams with all external lines outside of the circle, such as the one in figure 1, will be referred to as leading N_c MHV diagrams, since they give the $G_{n;1}$ color factors; those with external lines both outside and inside the circle, such as the one in figure 8, as sub-leading MHV diagrams, since they give the $G_{n;c>1}$ color factors. (One-loop MHV diagrams with all external lines inside of the circle are identical to leading MHV diagrams, up to possible overall signs.) As indicated above, the former give the leading amplitudes $A_{n;1}$. The latter should give the sub-leading amplitudes $A_{n;c>1}$ if the CSW prescription is independent of the large N_c limit. The proof of this assertion is the main focus of this paper.

3. Special cases of four and five external particles

In this section, we show that the CSW prescription gives the full one-loop MHV amplitudes in cases of four and five external particles.

We note that one cannot make a loop out of a single MHV vertex. Such diagrams would include the following η integration,

$$\int d^4\eta^l \delta^8(\Theta)\,. \tag{3.1}$$

However,

$$\Theta = \sum_{i=1}^{n} \eta_A^i \lambda_i^\alpha + \eta_A^l l^\alpha - \eta_A^l l^\alpha = \sum_{i=1}^{n} \eta_A^i \lambda_i^\alpha \tag{3.2}$$

which is independent of η^l. The η integral thus yields a null result. The scattering amplitudes of n gluons with one or none opposite type of helicity also vanish, as expected. One-loop MHV diagrams are constructed from two and only two MHV vertices.

Next we prove that one-loop MHV diagrams with three-leg MHV vertices also give null results. We start with a simple diagram with all external lines outside of the circle, as shown in figure 2. After integrating out the fermionic variables, this diagram gives

$$\int \frac{d^4L_1}{L_1^2}\frac{d^4L_2}{L_2^2}\delta^4(L_1+L_2+P_\lambda)\langle l_1, l_2\rangle^2 F \tag{3.3}$$

where F is a complicated factor but irrelevant to our discussion. Due to energy-momentum conservation, $L_1 + L_2 + P_\lambda = 0$, the off-shell spinors are related

$$\begin{aligned}(l_1)_\alpha &= -(L_2)_{\alpha\dot\alpha}\tilde\xi^{\dot\alpha} - \lambda_{\alpha\dot\alpha}\tilde\xi^{\dot\alpha} \\ &= -(l_2)_\alpha - \lambda_{\alpha\dot\alpha}\tilde\xi^{\dot\alpha}\end{aligned} \tag{3.4}$$

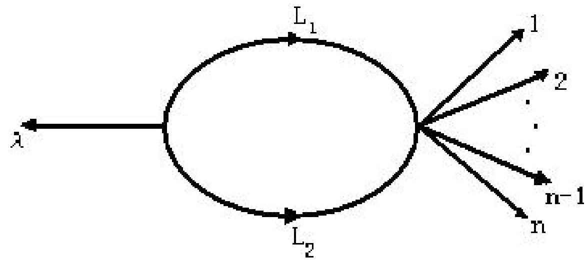

Figure 2: A one-loop MHV diagram with one three-leg MHV vertex, shown to give a null result.

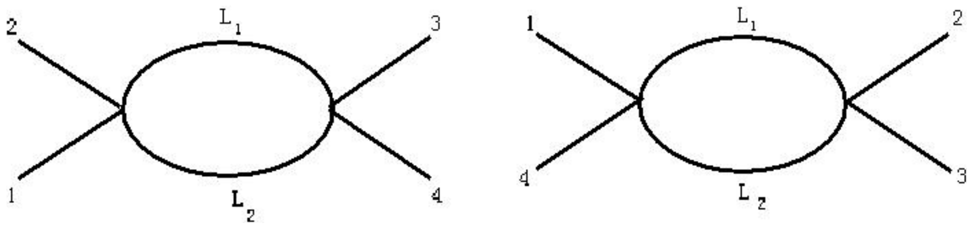

Figure 3: (a) Leading one-loop MHV diagrams for the special case of four external particles.

up to irrelevant factors. Since the integration in eq. (3.3) is independent of $\tilde{\xi}$ [19], we can choose $\tilde{\xi}^{\dot{\alpha}} = \tilde{\lambda}^{\dot{\alpha}}$. We have then $l_1 = -l_2$ so the integrand in eq. (3.3) vanishes. For general diagrams with external lines both inside and out side of the circle, a theorem to be proved in next section states that they can be expressed as sums of diagrams with all particles outside of the circle. This closes our proof.

Now we consider one-loop MHV diagrams of four external particles. They are made of two MHV vertices and each vertex has four legs. The two diagrams in figure 3a give the leading contribution with color factor $N_c \, \mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$

$$A_{4;1}(1,2,3,4) = \int \frac{d^4L_1}{L_1^2}\frac{d^4L_2}{L_2^2} F_0(1,2,3,4) \tag{3.5}$$

where

$$F_0(1,2,3,4) \;=\; \delta^4(L_1+L_2+P_{1L})h_1(1,2,3,4) + \delta^4(L_1+L_2+P_{2L})h_2(1,2,3,4) \tag{3.6}$$

with $P_{1L} = \lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2$, $P_{2L} = \lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4$ and

$$\begin{aligned} h_1(1,2,3,4) &= \frac{1}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,1\rangle}\frac{\langle 2,3\rangle\langle l_2,l_1\rangle}{\langle 2,l_1\rangle\langle l_1,3\rangle}\frac{\langle 4,1\rangle\langle l_1,l_2\rangle}{\langle 4,l_2\rangle\langle l_2,1\rangle} \\ h_2(1,2,3,4) &= \frac{1}{\langle 4,1\rangle\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle}\frac{\langle 1,2\rangle\langle l_2,l_1\rangle}{\langle 1,l_1\rangle\langle l_1,2\rangle}\frac{\langle 3,4\rangle\langle l_1,l_2\rangle}{\langle 3,l_2\rangle\langle l_2,4\rangle} \end{aligned} \tag{3.7}$$

eq. (3.5) gives the correct leading contribution.

We then turn to the sub-leading terms. For illustration, we take the terms with the color factor $\mathrm{Tr}\,(T^{a_1}T^{a_2})\,\mathrm{Tr}\,(T^{a_3}T^{a_4})$. Terms with other color factors can be obtained by simple permutation of color indices and corresponding helicities and momenta. By using eq. (2.6), we can construct $A_{4;3}(1,2;3,4)$ in terms of the $A_{4;1}$ in eq. (3.5). This gives a similar integral as eq. (3.5) but with the following integrand:

$$F_0(1,2,3,4)+F_0(1,3,2,4)+F_0(3,1,2,4)+F_0(2,1,3,4)+F_0(2,3,1,4)+F_0(3,2,1,4) \tag{3.8}$$

Now the sub-leading MHV diagrams. There are six of them in total (figure 3b), each gives a similar integral as eq. (3.5) but with a different h and related P_L:

$$\hat{h}_1(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle 4,l_1\rangle\langle l_1,1\rangle\langle 1,l_2\rangle\langle l_2,4\rangle\langle l_1,3\rangle\langle 3,l_2\rangle\langle l_2,2\rangle\langle 2,l_1\rangle} \tag{3.9}$$

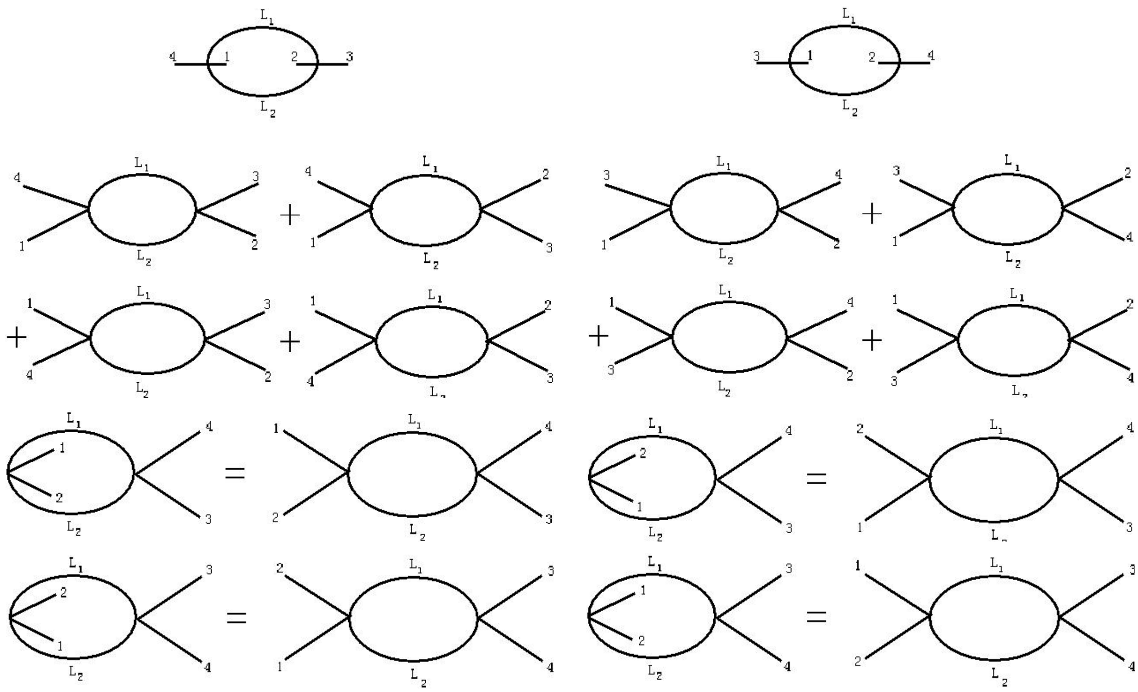

Figure 3: (b) Sub-leading one-loop MHV diagrams for the special case of four external particles.

with $P_L = \lambda_1\tilde{\lambda}_1 + \lambda_4\tilde{\lambda}_4$;

$$\hat{h}_2(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle 3,l_1\rangle\langle l_1,1\rangle\langle 1,l_2\rangle\langle l_2,3\rangle\langle l_1,4\rangle\langle 4,l_2\rangle\langle l_2,2\rangle\langle 2,l_1\rangle} \tag{3.10}$$

with $P_L = \lambda_1\tilde{\lambda}_1 + \lambda_3\tilde{\lambda}_3$; and

$$\hat{h}_3(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle l_1,1\rangle\langle 1,2\rangle\langle 2,l_2\rangle\langle l_2,l_1\rangle\langle l_1,4\rangle\langle 4,3\rangle\langle 3,l_2\rangle\langle l_2,l_1\rangle} \tag{3.11}$$

$$\hat{h}_4(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle l_1,2\rangle\langle 2,1\rangle\langle 1,l_2\rangle\langle l_2,l_1\rangle\langle l_1,4\rangle\langle 4,3\rangle\langle 3,l_2\rangle\langle l_2,l_1\rangle} \tag{3.12}$$

$$\hat{h}_5(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle l_1,2\rangle\langle 2,1\rangle\langle 1,l_2\rangle\langle l_2,l_1\rangle\langle l_1,3\rangle\langle 3,4\rangle\langle 4,l_2\rangle\langle l_2,l_1\rangle} \tag{3.13}$$

$$\hat{h}_6(1,2;3,4) = \frac{\langle l_1,l_2\rangle^4}{\langle l_1,1\rangle\langle 1,2\rangle\langle 2,l_2\rangle\langle l_2,l_1\rangle\langle l_1,3\rangle\langle 3,4\rangle\langle 4,l_2\rangle\langle l_2,l_1\rangle} \tag{3.14}$$

all with $P_L = \lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2$. With the help of Schouten identity, we have

$$\begin{aligned}
\hat{h}_1(1,2;3,4) &= h_1(1,4,3,2) + h_1(1,4,2,3) + h_2(1,3,2,4) + h_2(1,2,3,4)\,,\\
\hat{h}_2(1,2;3,4) &= h_1(1,3,4,2) + h_1(1,3,2,4) + h_2(1,4,2,3) + h_2(1,2,4,3)\,,\\
\hat{h}_3(1,2;3,4) &= h_2(1,4,3,2)\,,\\
\hat{h}_4(1,2;3,4) &= h_1(1,2,4,3)\,,\\
\hat{h}_5(1,2;3,4) &= h_1(1,2,3,4)\,,\\
\hat{h}_6(1,2;3,4) &= h_2(1,3,4,2)\,,
\end{aligned} \tag{3.15}$$

all graphically illustrated in figure 3b. Putting all terms together, one gets exactly the sum of F's in eq. (3.8).

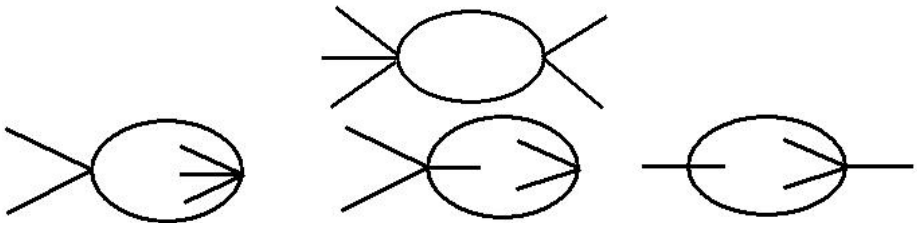

Figure 4: The topologies for leading (top) and sub-leading (bottom) one-loop MHV diagrams for the special case of five external particles.

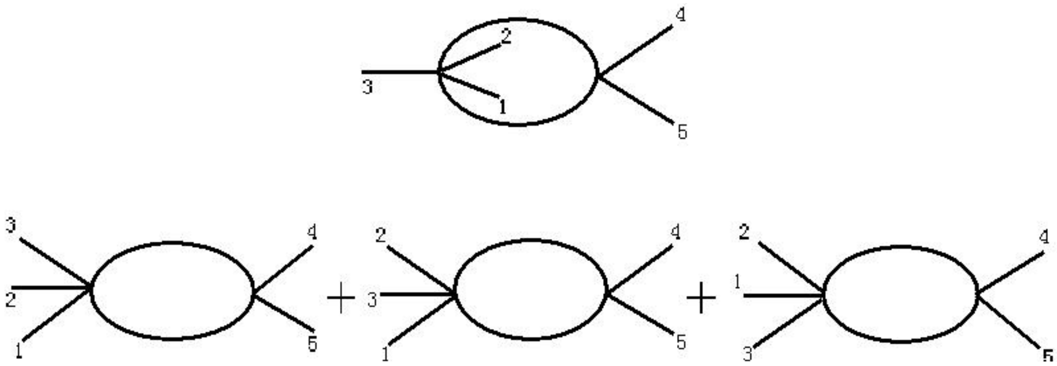

Figure 5: The decomposition of a sub-leading one-loop MHV diagram in terms of leading one-loop MHV diagrams for one case of five external particles.

For one-loop MHV diagrams of five external particles, one vertex has four legs and the other has five legs. For a given color factor, $\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})$, there is only one topology, as shown on the top of figure 4, but five different MHV diagrams. These five diagrams combine to give a total integrand $F_0(1,2,3,4,5)$ and $A_{5;1}(1,2,3,4,5)$ after internal momentum integration.

For the sub-leading term with a given color factor, $\mathrm{Tr}\,(T^{a_1}T^{a_2})\,\mathrm{Tr}\,(T^{a_3}T^{a_4}T^{a_5})$, we can similarly construct $A_{5;3}$ out of $A_{5;1}$. This gives an integral with the following integrand,

$$\begin{aligned}&F_0(2,1,3,4,5)+F_0(2,3,1,4,5)+F_0(2,3,4,1,5)+F_0(3,2,1,4,5)+\\&+F_0(3,2,4,1,5)+F_0(3,4,2,1,5)+F_0(1,2,3,4,5)+F_0(1,3,2,4,5)+\\&+F_0(1,3,4,2,5)+F_0(3,1,2,4,5)+F_0(3,1,4,2,5)+F_0(3,4,1,2,5)\,.\end{aligned}\tag{3.16}$$

Now the sub-leading MHV diagrams. There are three topologies, as shown at the bottom of figure 4, but sixty different MHV diagrams. Each can be re-arranged as sum of diagrams in the leading term, after proper permutation of indices. Shown in figure 5 is one of the re-arrangement. After a straightforward but lengthy calculation, the sixty diagrams are shown to yields the same expression in eq. (3.16). So in both cases, sub-leading MHV diagrams do give the correct sub-leading amplitudes.

4. General cases of arbitrary number of external particles

In this section, we show that the CSW prescription gives the correct full one-loop MHV amplitudes in the general case of arbitrary number of external particles. To fix notation, we first give the prescription to obtain the leading contribution. For a given color factor, $\mathrm{Tr}\,(T^{a_1}\cdots T^{a_n})$, one gets all MHV diagrams from figure 1, by distributing external particles on the two vertices in all possible manners but keeping the relative order according to the color indices. Specifically, we first put $m = \{1, \dots [n/2]\}$ external lines on the first vertex and the rest on the second one. As shown in the last section, the $m = 1$ cases give a null result. But they are kept here, as our results do not depend on the mentioned fact. For each m, we put on the external lines clock-wise according to the color ordering to get a diagram, and use cyclic permutation to get all different ones:

$$A_{n;1} = \sum_m \int \frac{d^4L_1}{L_1^2}\frac{d^4L_1}{L_1^2} \sum_{\sigma\in(\{\alpha\})} \delta^{(4)}(L_1+L_2+P_L)F(m,\sigma) \tag{4.1}$$

where

$$F(m,\sigma) = \frac{\langle\sigma(m),\sigma(m+1)\rangle\langle\sigma(n),\sigma(1)\rangle\langle l_1,l_2\rangle\langle l_2,l_1\rangle}{\langle l_2,\sigma(1)\rangle\langle\sigma(m),l_1\rangle\langle l_1,\sigma(m+1)\rangle\langle\sigma(n),l_2\rangle}\prod_{i=1}^{n}\frac{1}{\langle\sigma(i),\sigma(i+1)\rangle} \tag{4.2}$$

P_L is the total momentum of all external particles on the left MHV vertex, and $(\{\alpha\})$ is the set of all cyclic permutation of n objects. $A_{n;1}$ is identical to the ones obtained by conventional quantum field theory calculation [19].

For a specific sub-leading term with the color factor

$$\mathrm{Tr}\,(T^{a_1}\cdots T^{a_{c-1}})\,\mathrm{Tr}\,(T^{a_c}\cdots T^{a_n}) \tag{4.3}$$

we construct the sub-leading term $A_{n;c}(1,\dots,c-1;c,\dots,n)$ according to eq. (2.6). A lengthy but straightforward calculation gives

$$A_{n;c} = \sum_m \int \frac{d^4L_1}{L_1^2}\frac{d^4L_2}{L_2^2}(-1)^{c-1} \sum_{\sigma\in COP''\{\alpha\}\{\beta\}} \delta^{(4)}(L_1+L_2+P_L)F(m,\sigma) \tag{4.4}$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1,\dots,1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \dots, n\}$ and $COP''\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,\dots,n\}$ that preserve the cyclic ordering of α_i within $\{\alpha\}$ and of β_i within $\{\beta\}$, while allowing for all possible relative ordering of α_i with respect to β_i.

In the rest of the section, we will show that this $A_{n;c}(1,\dots,c-1;c,\dots,n)$ can also be obtained from all relevant sub-leading MHV diagrams. Explicitly, these sub-leading MHV diagrams are generated by distributing the first $c-1$ external particles inside the circle, and the rest $n-c+1$ external particles outside the circle, but keeping the cyclic order inside and outside of the circle, respectively.

Before starting, we note that the following relation between MHV amplitudes

$$\sum_{k=1}^{n-1}\left(\prod_{i=1}^{k-1}\frac{1}{\langle i,i+1\rangle}\right)\frac{1}{\langle k,\lambda\rangle\langle\lambda,k+1\rangle}\left(\prod_{i=k+1}^{n}\frac{1}{\langle i,i+1\rangle}\right) = -\frac{1}{\langle n,\lambda\rangle}\frac{1}{\langle\lambda,1\rangle}\prod_{i=1}^{n-1}\frac{1}{\langle i,i+1\rangle} \tag{4.5}$$

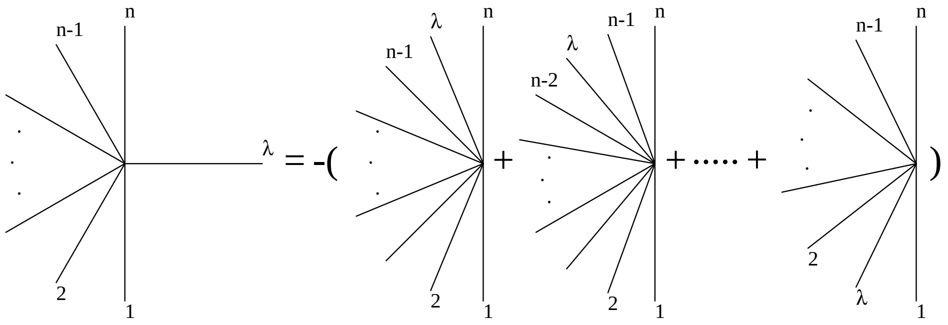

Figure 6: A graphic representation of eq. (4.5), one MHV amplitude is expressed in terms of other MHV amplitudes. On the left-side of the equation, line λ is to the right of lines 1 and n. On the right-side of the equation, line λ is reflected to the left of lines 1 and n, and in all possible positions among the lines originally on the left.

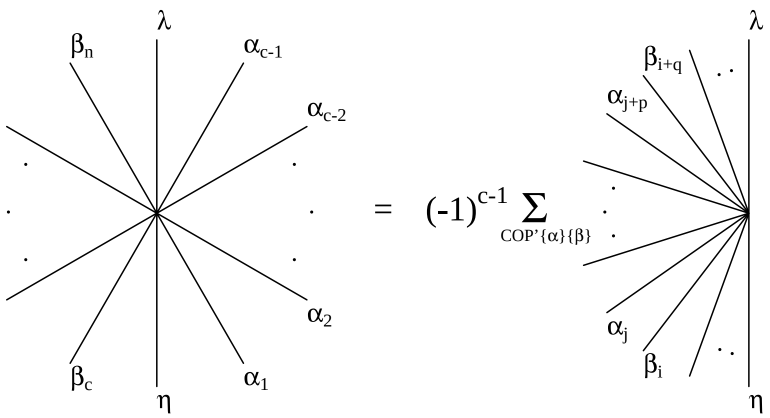

Figure 7: Graphic representation of eq. (4.6), another way to express one MHV amplitude in terms of other MHV amplitudes. On the left-side of the equation, all α lines are to the right of lines λ and η. On the right-side of the equation, all α lines are reflected to the left of lines λ and η, and in all possible positions among the lines originally on the left, while keeping the relative orders among themselves.

which can be proved in a straightforward manner by repeating use of the Schouten identity. Shown in figure 6 is a graphic representation of the identity. By repeating use of this relation, one further gets

$$\frac{1}{\langle\lambda,\alpha_{c-1}\rangle\langle\alpha_1,\eta\rangle\langle\eta,\beta_c\rangle\langle\beta_n,\lambda\rangle}\left(\prod_{i=c-1}^{2}\frac{1}{\langle\alpha_i,\alpha_{i-1}\rangle}\right)\left(\prod_{j=c}^{n-1}\frac{1}{\langle\beta_j,\beta_{j+1}\rangle}\right)=$$

$$=(-1)^{c-1}\frac{1}{\langle\lambda,\eta\rangle}\sum_{\sigma\in COP'\{\alpha\}\{\beta\}}\frac{1}{\langle\eta,\sigma(1)\rangle}\left(\prod_{i=1}^{n-1}\frac{1}{\langle\sigma(i),\sigma(i+1)\rangle}\right)\frac{1}{\langle\sigma(n),\lambda\rangle}\tag{4.6}$$

where $\alpha_i\in\{\alpha\}\equiv\{c-1,\ldots,1\}$, $\beta_i\in\{\beta\}\equiv\{c,c+1,\ldots,n\}$ and $COP'\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1,\ldots,n\}$ that preserve the ordering of α_i within $\{\alpha\}$ and of β_i within $\{\beta\}$, while allowing for all possible relative ordering of α_i with respect to β_i. figure 7 is a graphic representation of this identity.

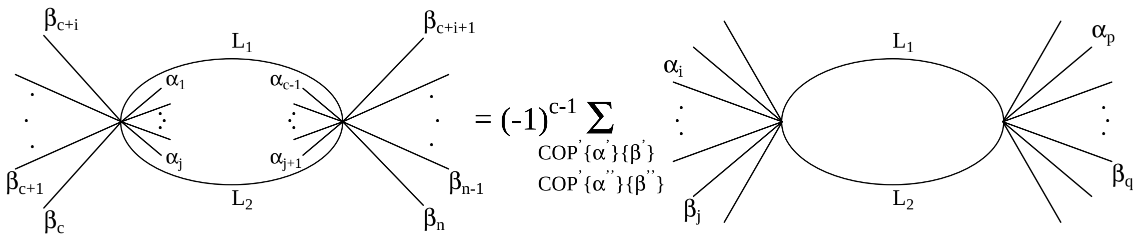

Figure 8: Graphic representation of eq. (4.7), a one-loop sub-leading MHV diagram is expressed in terms of a set of one-loop leading MHV diagrams. All the external lines inside the circle are reflected to outside, in the manner prescribed in eq. (4.6) and figure 7.

We now take a specific diagram with m external lines on the first vertex, among them, j of which inside the circle and $i+1$ of which outside of the circle, $m = i+j+1$, as shown on the left of figure 8. Applying eq.(4.6), we can make the following transformation for graphs on the left side of figure 8 to its right side:

$$\begin{aligned}
&\left(\prod_{r=c}^{c+i-1} \frac{1}{\langle \beta_r, \beta_{r+1}\rangle}\right) \frac{1}{\langle \beta_{c+i}, l_1\rangle\langle l_1, \alpha_1\rangle} \left(\prod_{r=1}^{j-1} \frac{1}{\langle \alpha_r, \alpha_{r+1}\rangle}\right) \frac{1}{\langle \alpha_j, l_2\rangle\langle l_2, \beta_c\rangle}\times \\
&\times \left(\prod_{r=j+1}^{c-2} \frac{1}{\langle \alpha_r, \alpha_{r+1}\rangle}\right) \frac{1}{\langle \alpha_{c-1}, l_1\rangle\langle l_1, \beta_{c+i+1}\rangle} \left(\prod_{r=c+i+1}^{n-1} \frac{1}{\langle \beta_r, \beta_{r+1}\rangle}\right) \frac{1}{\langle \beta_n, l_2\rangle\langle l_2, \alpha_{j+1}\rangle} = \\
&= (-1)^{c-1} \left[\frac{1}{\langle l_1, l_2\rangle} \sum_{\sigma \in COP'\{\alpha'\}\{\beta'\}\}} \frac{1}{\langle l_2, \sigma(1)\rangle} \left(\prod_{i=1}^{m-1} \frac{1}{\langle \sigma(i), \sigma(i+1)\rangle}\right) \frac{1}{\langle \sigma(m), l_1\rangle}\right] \times \\
&\times \left[\frac{1}{\langle l_2, l_1\rangle} \sum_{\sigma \in COP'\{\alpha''\}\{\beta''\}} \frac{1}{\langle l_1, \sigma(1)\rangle} \left(\prod_{i=1}^{n-m+1} \frac{1}{\langle \sigma(i), \sigma(i+1)\rangle}\right) \frac{1}{\langle \sigma(n-m), l_2\rangle}\right]
\end{aligned} \tag{4.7}$$

where $\alpha_i \in \{\alpha'\} \equiv \{j, \ldots, 1\}$ or $\{\alpha''\} \equiv \{c-1, \ldots, j+1\}$, $\beta_i \in \{\beta'\} \equiv \{c, \ldots, c+i\}$ or $\in \{\beta''\} \equiv \{c+i+1, \ldots, n\}$, $COP'\{\alpha'\}\{\beta'\}$ ($COP'\{\alpha''\}\{\beta''\}$) is the set of all permutations of $\{1, \ldots, m\}$ ($\{m+1, \ldots, n\}$) that preserve the ordering of α_i within $\{\alpha'\}$ ($\{\alpha''\}$) and of β_i within $\{\beta'\}$ ($\{\beta''\}$), while allowing for all possible relative ordering of α_i with respect to β_i. Cyclically permuting all elements in $\{\alpha\} = \{\alpha'\} \bigcup \{\alpha''\}$ and $\{\beta\} = \{\beta'\} \bigcup \{\beta''\}$, and take the sum for all cases of i, j and m, we get

$$\sum_m (-1)^{c-1} \sum_{\sigma \in COP''\{\alpha\}\{\beta\}} \delta^{(4)}(L_1 + L_2 + P_L) F(m, \sigma) \tag{4.8}$$

which is nothing but the integrand in eq. (4.4).

Thus we have proved our main assertion that sub-leading amplitudes $A_{n;c}$ obtained from the sub-leading MHV diagrams are related to the leading N_c amplitudes $A_{n;1}$ in the same way as those obtained from conventional field theory calculations. The CSW prescription gives the correct prediction for MHV amplitudes to one loop without taking the large N_c limit.

5. Conclusion

In this paper, we have analyzed the MHV amplitudes in general $N = 4$ super Yang-Mills theories to full one-loop by using the CSW prescription. We reproduce the same relation between leading N_c amplitudes $A_{n;1}$ and sub-leading amplitudes $A_{n;c}$ as those obtained from conventional field theory calculations [2]. Combining with existing results [19], this establishes the validity of the CSW approach to one-loop in the calculation of MHV amplitudes, without taking the large N_c limit.

It is tempting to speculate that the method is valid to non-MHV diagrams and to all orders, though an understanding of the approach from the perspective of conventional quantum field theory is still wanting. From the viewpoint of quantum field theory, the CSW prescription could be taken as an efficient organization principle of Feynman diagrams. It is motivated by a string construction and the string construction so far seems mainly to be related to the large N_c limit. The sub-leading results obtained in this paper indicate a wider applicability. This calls for further inquiry of the rationale behind the prescription. Parallel to the original motivation, we believe that a construction of the prescription from conventional field theory is feasible, as well as highly desirable, given the highly restrictive nature of $N = 4$ SYM theories. Such a construction may also change our perception of perturbative quantum field theories with enormous symmetries. In any case, it will be interesting to extend the analysis to non-MHV diagrams as well as to higher order calculations.

Acknowledgments

This work is supported in part by the National Science Foundation of China.

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